Introduction to Mathematics Through Calculus: Chapter One

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The Differential:

Throughout Calculus, we will say that the notation

$$d(x) = dx$$

is used to denote a really small (*infinitesimal*) change in x. *d(x)* is called *the differential of x*, and can be assumed to be some kind of a function. Left alone, a differential is said to be equal to zero; however, the ratio of two differentials do make sense, as they are of relative order. All in all, we can think of differentials as quantities, or sequences, tending to zero. We can manipulate them like regular functions and numbers, and get rid of them by taking a *limit*; that is, making them approach to a number. This number, as you can imagine, is, in this case, zero.

Derivative:

A derivative can be said to be the ratio of the aforementioned differential (change) between two related variables.
Let us consider the function

$$f(x) = x$$

We want to find the derivative of f(x) with respect to x, that is:

$$\frac{df(x)}{dx}$$

Now, let us add the differential terms to the function above:

$$f(x) + df(x) = x + dx$$

Since x equals f(x), the two terms cancel out, leaving

$$df(x) = dx$$

And

$$\frac{df(x)}{dx} = 1$$

We have successfully found our first derivative.

Now, let us consider the function

$$f(x) = x^2$$

Add the differential terms:

$$f(x) + df(x) = (x + dx)^2$$

Expand the right hand side:

$$f(x) + df(x) = x^2 + 2xdx + (dx)^2$$

Cancel out the terms we know are equal. Now we are left with

$$df(x) = 2xdx + (dx)^2$$

Divide both sides with dx to get

$$\frac{df(x)}{dx} = 2x + dx$$

We now have the derivative at the left hand side. Now we take the limit, and reduce the dx term to zero. Thus, we get

$$\frac{df(x)}{dx} = 2x$$

The common notation used for a derivative in mathematics is

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

This is just a symbolic presentation what we've been doing.

Example:

Let us consider the function

$$f(x) = \frac{1}{x}$$

Now we add the differentials:

$$f(x) + df(x) = \frac{1}{x + dx}$$

We subtract f(x) from both sides and replace it with 1/x:

$$df(x) = \frac{1}{x + dx} - \frac{1}{x} = \frac{-dx}{x^2 + xdx}$$

We divide both sides by dx and reduce the remaining differential to zero to get

$$\frac{df(x)}{dx} = \frac{-1}{x^2}$$

Product Rule:

Let us define the function f(x) as

$$f(x) = g(x)h(x)$$

And let's say that we want to find

$$\frac{df(x)}{dx}$$

Let us add the differential terms:

$$f(x) + df(x) = [g(x) + dg(x)][h(x) + dh(x)]$$
$$= g(x)h(x) + gdh(x) + hdg(x) + dg(x)dh(x)$$

Subtracting the initial equation, dividing by dx, and reducing the remaining differentials to zero, we get

$$\frac{df(x)}{dx} = g(x)\frac{dh(x)}{dx} + h(x)\frac{dg(x)}{dx}$$

This is called the product rule.

Chain Rule:

In general, we can say that

$$\frac{df(x)}{dx} = \frac{df(x)}{dg(x)}\frac{dg(x)}{dx}$$

This simple rule of taking derivatives is called the chain rule.

Partial Differentiation:

Let us define a function f that depends on two variables x and y

$$f(x, y) = xy$$

Then,

$$\frac{df(x, y)}{dx} = y + x\frac{dy}{dx}$$

Now, we define the partial derivative as

$$\frac{\partial f(x, y)}{\partial x} = y$$

In general, we can say that in partial differentiation, all variable differentials except the one we are taking the derivative relative to are equal to zero.

Integration:

Integration is the reverse process of derivation. Note that

$$\frac{df(x)}{dx} = \frac{df(x)}{dx} \Rightarrow df(x) = \frac{df(x)}{dx}\ dx$$

The common notation for integration is as follows:

$$\int \frac{df(x)}{dx}dx = \int df(x) = f(x) + c$$

c is used to denote a constant; it is there because, in general, if *f(x) =c*, then the derivative of f(x) with respect to x is 0, because *dc* is always zero.

We can always say that the integral sign is the inverse of the differential.

Example:

$$\int 3xdx = ?$$

We know that

$$\frac{d(x^2)}{dx} = 2x$$

And that, as such,

$$\int 3xdx = \int \frac{3}{2}\frac{dx^2}{dx}dx = \int \frac{3}{2}dx^2 = \frac{3}{2}x^2 + c$$

Integration by Parts:

Let us look at the product rule again:

$$\frac{d(g(x)h(x))}{dx} = g(x)\frac{dh(x)}{dx} + h(x)\frac{dg(x)}{dx}$$

Multiply every term by dx and rearrange:

$$d\big(g(x)h(x)\big) = g(x)dh(x) + h(x)dg(x)$$

$$g(x)dh(x) = d\big(g(x)h(x)\big) - h(x)dg(x)$$

Integrating, we will get:

$$\int g(x)dh(x) = g(x)h(x) - \int h(x)dg(x)$$

This is called integration by parts.

The Constant e:

Let us consider the function

$$f(x) = e^x$$

In which e is a constant. Let us try to find the derivative:

$$f(x) + df(x) = e^{x+dx} = e^x e^{dx} = f(x)e^{dx}$$

$$df(x) = f(x)(e^{dx} - 1)$$

This equality is of immense importance to mathematics. Note that, if

$$\left(e^{dx} - 1\right) = dx$$

We will get the function for which

$$\frac{df(x)}{dx} = f(x)$$

The mathematical constant *e* is defined as follows:

$$e = \ (dx + 1)^{\frac{1}{dx}}$$

A common notation for *e* using the limit notation is

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

However,

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n^2}\right)^{n^2}$$

Hence, our notation allows for a more general form of the constant *e*.
Now, we know that

$$\frac{de^x}{dx} = e^x$$

The Natural Logarithm:
Let us consider the function

$$f(x) = e^x$$

We now introduce the function ln(x), the natural logarithm, as follows:

$$\ln(e^x) = x$$

In general, we can check to see that the following equality holds:

$$\ln(g(x)h(x)) = \ln\big(g(x)\big) + \ln\,(h(x))$$

Example:
We want to find the derivative of the function

$$\mathrm{f(x)} = \ln\,(x)$$

We take the exponential of both sides:

$$x = e^{f(x)}$$

Now we take the derivative of both sides, this time applying what we have learned so far:

$$1 = e^{f(x)}\frac{df(x)}{dx}$$

The last term in the right hand side came from **the chain rule**.

$$\frac{1}{e^{f(x)}} = \frac{df(x)}{dx}$$

Replacing the denominator in the left hand side, we get

$$\frac{df(x)}{dx} = \frac{1}{x}$$

Imaginary Numbers:
We define the number *i* as:

$$i = \sqrt{-1}$$

Consequently:

$$i^2 = -1$$
$$i^4 = 1$$

That's all we need to know for now.

Sine, Cosine and Tangent:

We define two functions, sin(x) and cos(x) as follows:

$$f(x) = \sin\,(x)$$

$$\frac{df(x)}{dx} = \cos\,(x)$$

Additionally,

$$\sin(x)^2 + \cos(x)^2 = 1$$

Is the Pythagorean trigonometric identity. We can prove this theorem, quite easily, using differentiation, or by considering the unit circle and defining sine and cosine on it.

Additional derivatives of the function f(x) = sin(x) are as follows:

$$\frac{d\frac{df(x)}{dx}}{dx} = \frac{d^2f(x)}{dx^2} = -\sin(x)$$

$$\frac{d^3f(x)}{dx^3} = -\cos(x)$$

$$\frac{d^4f(x)}{dx^4} = \sin(x)$$

Finally, we define tan(x) as

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Example:

We want to find the derivative of the function

$$\mathrm{f(x)} = \tan(x)$$

We will be using both the product rule and the chain rule for this function.

$$\mathrm{f(x)} = \tan(x) = \frac{\sin(x)}{\cos(x)} = \sin(x)\frac{1}{\cos(x)}$$

Now, we take the derivative:

$$\frac{\mathrm{df(x)}}{\mathrm{dx}} = \cos(x)\frac{1}{\cos(x)} - \sin(x)\frac{1}{\cos(\mathrm{x})^2} * -\sin(x) = 1 + \tan(x)^2 = \frac{\sin(\mathrm{x})^2 + \cos(\mathrm{x})^2}{\cos(\mathrm{x})^2} = \frac{1}{\cos(\mathrm{x})^2}$$

Euler's Formula:

Let us consider the function

$$f(x) = \mathrm{A} * \cos(\mathrm{x}) + \mathrm{B} * \sin(\mathrm{x})$$

where A and B are constants.

Now, we can see that

$$\frac{d^2f(x)}{dx^2} = -A * \cos(x) - \mathrm{B} * \sin(x) = -f(x)$$

Now, we know what f(x) is.

$$f(x) = e^{ix}$$

As such, we now know the identity

$$e^{ix} = A * \cos(\mathrm{x}) + \mathrm{B} * \sin(\mathrm{x})$$

Holds for some values A and B.

We can also see that

$$\frac{df(x)}{dx} = -\mathrm{A} * \sin(\mathrm{x}) + \mathrm{B} * \cos(\mathrm{x}) = \mathrm{i}e^{ix}$$

Which lets us see that A=1 and B=i, giving us the final form of Euler's Formula

$$e^{ix} = \cos(\mathrm{x}) + \mathrm{i}\sin(\mathrm{x})$$

Cauchy-Riemann Equations:

Let us consider the equation

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

Now, we add the differential terms, taking dy=0,

$$f(z) + df(z) = df(x) = u(x, y) + du(x, y) + iv(x, y) + idv(x, y)$$

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

We use partial differentiation because we took dy to be 0. Now, we do the same thing again, but this time taking dx=0,

$$f(z) + df(z) = df(iy) = idf(y) = u(x, y) + du(x, y) + iv(x, y) + idv(x, y)$$

$$\frac{\partial f(z)}{\partial y} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Obviously, *if the derivative is to exist*, these two partial derivatives should be equal. For them to be equal, the real parts and imaginary parts should be independently equal each other. Thus, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These two equations are called the **Cauchy-Riemann equations**.